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Centre For Particle Theory

DTP-89/39

Self-Dual Quaternionic Lumps in Octonionic Space-Time

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4 August 1989

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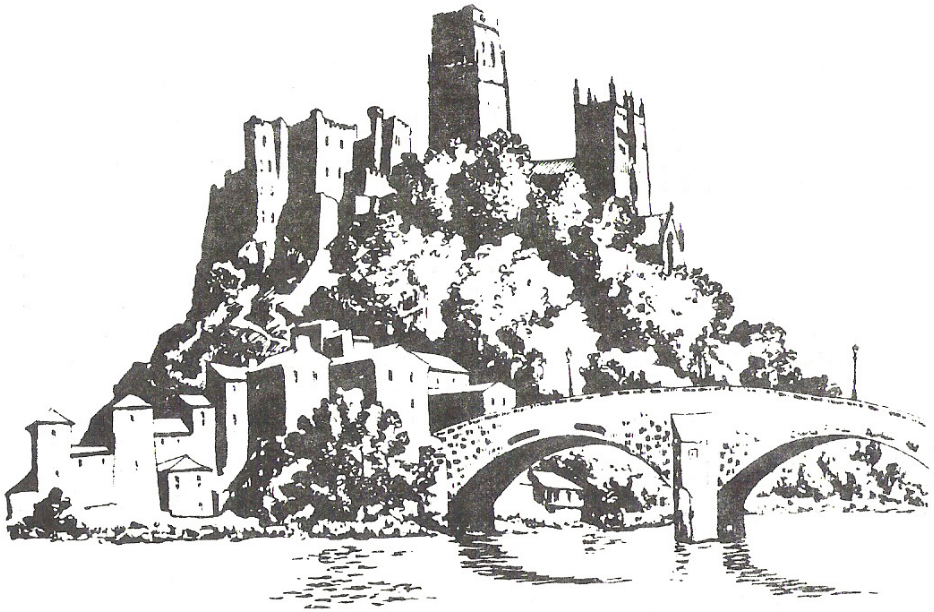

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0. Introduction

Moving from strings to p-branes (see [1]) has given extra vitality to discussion of the geometry of minimal immersions. The second order equations of motion of p-branes are in general highly non-linear and as such are hard to solve. However, Biran, Floratos and Savvidy [2] have pioneered an approach for constructing self-dual equations which are first order and are soluble in particular cases.

Recently, Grabowski and Tze [3] have pointed out that there might be a new class of exceptional geometries for which self-dual equations can be constructed. This paper concentrates on the case of a 3-brane (lump) in 8 dimensions which we call a (4;8)-brane.[1] We show that the self-duality ansatze does not automatically satisfy all the constraints, as was the case with the original example of Biran et al. and was said to be the case in [3], but rather we show that one of the constraints is equivalent to the self-dual equation itself.

By analogy with (p,q) torus knots invoked in the solution of self-dual (2;4)-brane [4], we introduce the quaternionic counterpart, (p,q) quots, as a proposal for the formulation of specific solutions of the self-dual (4;8)-brane equation. Although tantalizingly close, we show that some new idea is required before this ansatze will yield full solutions.

[1] The use of a semi-colon in this context is to distinguish our notation from that of some other authors who call this a (3,8)-brane.

1. Brane Dynamics

The action of a (d;D)-brane is given by the generalised Einstein-Nambu-Goto volume integral

$$S = T \int \sqrt{g}\; d^d\xi \tag{1.1}$$

where

$$g_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} \equiv X^\mu{}_{,a}\, X_{\mu,b} \tag{1.2}$$

with $\mu,\nu=0,1,\ldots,D-1$ and $a,b=0,1,\ldots,d-1$ unless D and d are odd in which case we count from 1 to D and 1 to d respectively. The fundamental constant of the theory, T, has dimensions $[ML^{1-d}]$, with the speed of light c=1.

Variation of this action gives the generalised geodesic equation

$$g^{ab}\left[\partial_a X^\mu{}_{,b} - \Gamma^c_{ab} X^\mu{}_{,c} + \Gamma^\mu_{\nu\lambda} X^\nu{}_{,a} X^\lambda{}_{,b}\right] = 0 \tag{1.3}$$

The (d;D)-brane is taken to be closed so that no boundary conditions need to be considered.

We consider the case of a flat Euclidean background space where $\eta_{\mu\nu}=$ Diag (1,1,...,1). We also introduce the conjugate momentum

$$P^{a\mu} \equiv \frac{\delta L}{\delta X_{\mu,a}} = T\sqrt{g}\; g^{ab} X^\mu{}_{,b} \tag{1.4}$$

The equation of motion (1.3) then reduces to

$$\partial_a P^{a\mu} = 0 \tag{1.5}$$

The constraints on the motion can be written

$$P^a{}_\mu X^\mu{}_{,b} = L\,\delta^a{}_b = T\sqrt{g}\;\delta^a{}_b \tag{1.6}$$

which corresponds to the vanishing of the Hamiltonian, and

$$P^a{}_\mu P^{b\mu} = T^2\, g\, g^{ab} \tag{1.7}$$

which is the extended object generalisation of $p^2=m^2$ for relativistic particles.

2. Self-Dual (d;d)-Branes

Consider (2;2)-brane (string in two dimensional space-time) and define self-dual (2;2)-brane by

$$\tilde{P}^a{}_\mu = T\, e^{ab} e_{\mu\nu} X^\nu{}_{,b} \tag{2.1}$$

where e^{ab} is the two dimensional permutation symbol fixed by

$$e^{01} \equiv +1 \tag{2.2}$$

Recall that the two dimensional permutation tensor ε^{ab} is defined by

$$\varepsilon^{ab} \equiv \frac{e^{ab}}{\sqrt{g}} \tag{2.3}$$

Also note that $\sqrt{\eta} = 1$ and therefore that

$$\varepsilon^{\mu\nu} = e^{\mu\nu} \tag{2.4}$$

in our particular case. $\tilde{P}^a{}_\mu$ is thus a worldsheet tensor density which satisfies (1.5) since $\partial_a X^\nu{}_{,b}$ is symmetric in a and b while e^{ab} is antisymmetric.

Expanding $\sqrt{g}$ explicitly for the (2;2)-brane gives

$$\sqrt{g} = X^0{}_{,0} X^1{}_{,1} - X^1{}_{,0} X^0{}_{,1} \tag{2.5}$$

while expanding $\tilde{P}^a{}_\mu X^\mu{}_{,b}$ gives

$$\tilde{P}^a{}_\mu X^\mu{}_{,b} = T e^{ac}\varepsilon_{\mu\nu} X^\nu{}_{,c} X^\mu{}_{,b} = T\begin{pmatrix} \left(X^0{}_{,0}X^1{}_{,1} - X^1{}_{,0}X^0{}_{,1}\right) & 0 \\ 0 & \left(X^1{}_{,1}X^0{}_{,0} - X^0{}_{,1}X^1{}_{,0}\right)\end{pmatrix} \tag{2.6}$$

showing that constraint (1.6) is satisfied. Substituting (2.1) into (1.7) gives

$$\begin{aligned}\tilde{P}^a{}_\mu \tilde{P}^{b\mu} &= T^2\, e^{ac}\varepsilon_{\mu\nu} X^\nu{}_{,c}\, e^{bd}\varepsilon^\mu{}_\rho X^\rho{}_{,d} \\ &= T^2\, e^{ac} e^{bd} g_{cd} = T^2 \,\mathrm{Adj}\; g_{ab} = T^2\, g\, g^{ab}\end{aligned} \tag{2.7}$$

So constraint (1.7) is satisfied. The self-dual equation is therefore $P^a{}_\mu = \tilde{P}^a{}_\mu$, that is, from (1.4) and (2.1),

$$X^\mu{}_{,a} = \varepsilon_a{}^b \varepsilon^\mu{}_\nu X^\nu{}_{,b} \tag{2.8}$$

Similarily it can be shown [5] that self-dual (d;d)-branes can be defined generally by

$$\tilde{P}^{a_1}{}_{\mu_1} = T \frac{1}{(d-1)!} e^{a_1 a_2 a_3 \dots a_d} \varepsilon_{\mu_1 \mu_2 \mu_3 \dots \mu_d} X^{\mu_2}{}_{,a_2} X^{\mu_3}{}_{,a_3} \dots X^{\mu_d}{}_{,a_d} \quad (2.9)$$

This satisfies the equation of motion by symmetry arguments. It also satisfies the constraints from the properties of contracted products of permutation tensors and by the definition of determinant in terms of permutation tensors.

The covariant self-dual (3;3)-brane equation is therefore

$$X^{\mu_1}{}_{,a_1} = \frac{1}{2} \varepsilon_{a_1}{}^{a_2 a_3} \varepsilon^{\mu_1}{}_{\mu_2 \mu_3} X^{\mu_2}{}_{,a_2} X^{\mu_3}{}_{,a_3} \quad (2.10)$$

In a gauge in which g_{ab} is diagonal, Biran et al. simplify (2.10) to

$$E^{\mu_1}{}_{a_1} = \frac{1}{2} e_{a_1 a_2 a_3} e^{\mu_1 \mu_2 \mu_3} E^{\mu_2}{}_{a_2} E^{\mu_3}{}_{a_3} \quad (2.11)$$

where

$$E^{\mu}{}_{a} \equiv \frac{X^{\mu}{}_{,a}}{\sqrt{X_{,a}{}^{2}}} \quad (2.12)$$

and show that

$$X(\xi) = R(\xi^1)\left\{\cos\Phi(\xi^3)\cos\Psi(\xi^2), \cos\Phi(\xi^3)\sin\Psi(\xi^2), \sin\Phi(\xi^3)\right\} \quad (2.13)$$

and

$$X(\xi) = \left\{\left[R+r(\xi^1)\cos\Phi(\xi^3)\right]\cos\Psi(\xi^2), \left[R+r(\xi^1)\cos\Phi(\xi^3)\right]\sin\Psi(\xi^2), r(\xi^1)\sin\Phi(\xi^3)\right\} \quad (2.14)$$

are solutions of (2.11).

$$X(\xi) = \left\{R(\xi^1)S(\xi^2)\cos\Theta(\xi^3), R(\xi^1)S(\xi^2)\sin\Theta(\xi^3), \frac{1}{2}\left[R(\xi^1)^2 - S(\xi^2)^2\right]\right\} \quad (2.15)$$

and

$$X(\xi) = \lambda\left\{\sinh\Theta(\xi^1)\sin\Phi(\xi^2)\cos\Psi(\xi^3), \sinh\Theta(\xi^1)\sin\Phi(\xi^2)\sin\Psi(\xi^3), \cosh\Theta(\xi^1)\cos\Phi(\xi^2)\right\} \quad (2.16)$$

are also solutions of (2.11).

3. Self-Dual (d;D)-Branes

The self-dual equation of rigid string instantons [4], self-dual (2;4)-branes, can be derived by defining

$$\tilde{P}^{a}{}_{\mu} = T e^{ab} J_{\mu\nu} X^{\mu}{}_{,b} \quad (3.1)$$

where an almost complex structure has been imposed on space-time by

$$J_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (3.2)$$

(3.1) automatically solves (1.5) and (1.7) but only satisfies (1.6) on condition that

$$\sqrt{g} = J_{\mu\nu} X^{\mu}{}_{,0} X^{\nu}{}_{,1} \quad (3.3)$$

as is the case if $X^{\mu}{}_{,a}$ has the symmetry of

$$X^{\mu}{}_{,a} \sim \begin{pmatrix} A & -B \\ B & A \\ C & D \\ -D & C \end{pmatrix} \quad (3.4)$$

which is true in, for example, the case of

$$X^{\mu} = \left[\, \tau^3 - 3\tau\sigma^2,\; 3\tau^2\sigma - \sigma^3,\; \sigma^2 - \tau^2,\; -2\tau\sigma \,\right] \quad (3.5)$$

where $\tau \equiv \xi^0$ and $\sigma \equiv \xi^1$.

Another interesting example of a self-dual (d;D)-brane (d<D) has been discussed by Grabowski and Tze [3]. This is the first exceptional case (see [6]), a (3;7)-brane (see also [7] for a (2;5)-brane). Define

$$\tilde{P}^{a}{}_{\mu} = T \frac{1}{2} e^{abc} C_{\mu\nu\lambda} X^{\nu}{}_{,b} X^{\lambda}{}_{,c} \quad (3.6)$$

where $C_{\mu\nu\lambda}$ are the octonion structure constants. (3.6) satisfies (1.5) because of the complete antisymmetry of $C_{\mu\nu\lambda}$. (3.6) also satisfies (1.7) as can be demonstrated using the identity

$$C_{\mu\nu\lambda} C^{\rho\sigma\lambda} = \delta_{\mu}{}^{\rho} \delta_{\nu}{}^{\sigma} - \delta_{\mu}{}^{\sigma} \delta_{\nu}{}^{\rho} + H_{\mu\nu}{}^{\rho\sigma} \quad (3.7)$$

where $\mu, \nu, \ldots = 1, 2, \ldots, 7$ and $H_{\mu\nu}{}^{\rho\sigma}$ is a completely antisymmetric rank 4 tensor which quantifies the non-associativity of octonion multiplication in a similar way that $C_{\mu\nu\lambda}$ quantifies their non-commutativity.

However, (1.6) is only satisfied if

$$\sqrt{g} = C_{\mu\nu\lambda} X^{\mu}{}_{,1} X^{\nu}{}_{,2} X^{\lambda}{}_{,3} \qquad (3.8)$$

Consider the self-dual (3;7)-brane equation resulting from (3.6) and (1.4),

$$X^{\mu}{}_{,a} = \frac{1}{2} \epsilon_{a}{}^{bc} C^{\mu}{}_{\nu\lambda} X^{\nu}{}_{,b} X^{\lambda}{}_{,c} \qquad (3.9)$$

Contracting (3.9) with $X_{\mu}{}^{,a}$ gives (3.8). The self-dual equation is satisfied if (3.8) is. The right side of (3.8) can be regarded as a generalised Jacobian for immersions.

One approach to obtaining a solution to (3.9) or (3.8) is to take the self-dual (3;3)-brane solutions (2.13-16) as trivial (3;7)-brane solutions. $C_{\mu\nu\lambda}$ is invariant under G_2 transformations. Taking a general matrix $G_{\mu\nu} \in G_2$, then

$$G_{\mu}{}^{\rho} G_{\nu}{}^{\sigma} G_{\lambda}{}^{\tau} C_{\rho\sigma\tau} = C_{\mu\nu\lambda} \qquad (3.10)$$

Substituting this into (3.8) gives

$$\sqrt{g} = C_{\rho\sigma\tau} \left[G_{\mu}{}^{\rho} X^{\mu}{}_{,1}\right] \left[G_{\nu}{}^{\sigma} X^{\nu}{}_{,2}\right] \left[G_{\lambda}{}^{\tau} X^{\lambda}{}_{,3}\right] \qquad (3.11)$$

Thus applying a general transformation of the seven dimensional representation of the exceptional Lie group G_2 to our trivial (3;7)-brane solutions will give the solutions in a form which which might be more interesting. We shall not pursue this approach here but we shall make a few more remarks about $C_{\mu\nu\lambda}$.

A basis for octonion multiplication is defined as *admissible* if it is such that

$$|O_1||O_2| = |O_1 O_2| \quad , \quad O_i \in \mathbb{O} \qquad (3.12)$$

which is equivalent to the condition that

$$\left[VecO_1 \wedge VecO_2\right]^2 = Det \left[VecO_i \, . \, VecO_j\right] \qquad (3.13)$$

where i,j=1,2. The caret symbol signifies the cross product in 7 dimensions which is defined by $C_{\mu\nu\lambda}$ (see [8]) and the dot signifies the 7 dimensional dot product. Both these products are a consequence of octonion multiplication. *VecO*, $O \in \mathbb{O}$, is similar to *ImZ*, $Z \in \mathbb{C}$, and selects the 7 dimensional vector part of *O*. (For a quaternion $Q \in \mathbb{H}$, *VecQ* selects the 3 dimensional vector part of *Q*.)

The identity

$$C_{\rho\sigma\tau} = \frac{1}{9} C_{\mu\rho}{}^{\nu} C_{\nu\sigma}{}^{\lambda} C_{\lambda\tau}{}^{\mu} \qquad (3.14)$$

identity,

$$C_{\rho\sigma\tau} = \frac{1}{4} H_{\rho\sigma}{}^{\mu\nu} C_{\mu\nu\tau} \qquad (3.15)$$

which follows by application of (3.7) to (3.14).

A particular basis is specified by a list of 7 triples each involving 3 of the integers 1 to 7 without repetition. Cyclic rotations of a triple give an equivalent triple. The quaternionic analogue is the basis for the usual cross product in 3 dimensions. This cross product is based on the SO(3) invariant tensor ϵ_{abc} which is characterised by a single triple involving the integers 1 to 3. The triple 123 is equivalent to 312 or 231 and specifies

$$\epsilon_{abc} = \sqrt{g}\; e_{abc} \qquad (3.16)$$

by defining $e_{123} \equiv +1$, which uniquely implies, by complete antisymmetry, all of the other components of e_{abc}. The only alternative basis for quaternions is the triple 132 which signifies the distinction between left and right handed coordinate systems in 3 dimensions.

For octonions there are 480 different choices of basis, 240 clockwise and 240 anti-clockwise. We call a basis *anti-clockwise* if it can be represented on the diagram in Figure 1 where A to G are to be identified one to one with the integers 1 to 7 in some order.

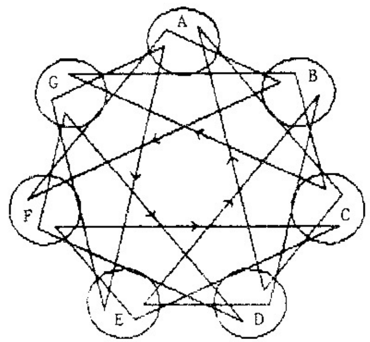

Figure 1. Representation of the 240 anti-clockwise bases of octonion multiplication.

Each of the 7 triangles represents one of the 7 triples, the arrow indicates the order. If, for example, A to G are identified with 1 to 7 respectively then figure 1 represents the

$$\text{basis}\quad \begin{vmatrix} 134 \\ 245 \\ 356 \\ 467 \\ 571 \\ 612 \\ 723 \end{vmatrix}.$$

This is interpreted as meaning that $C_{134}=1$ along with the other two even permutations, $C_{413}=C_{341}=1$. The three odd permutations are then -1, $C_{143}=C_{314}=C_{431}=-1$. This assignment process is repeated for each of the 7 triples and all other elements of $C_{\mu\nu\lambda}$ are set to zero. There are, on the face of it, 7! ways of placing 1 to 7 on figure 1. However the starting position is arbitrary so 7! should be divided by 7. Also since cyclic permutations are irrelevant, each basis is equivalent to two others generated by cyclically permuting the entire columns of the basis. Figure 1 therefore represents $\frac{7!}{7.3}= 240$ distinct bases. Changing the directions of all the arrows on figure 1 gives the 240 clockwise bases.

In summary, any admissible basis can be written such that each of the integers 1 to 7 appears one and only one time in each column, and each row does not have repetitions. The other important characteristic of an admissible basis is that no two rows have more than one integer in common. This rules out as

$$\text{inadmissible a basis such as}\quad \begin{vmatrix} 123 \\ 234 \\ 345 \\ 456 \\ 567 \\ 671 \\ 712 \end{vmatrix}.$$

4. Self-Dual (4;8)-Brane

Since quaternions have embeddings in octonions, $\mathbb{H} \hookrightarrow \mathbb{O}$, it is natural to generalise the (3;7)-brane, seen as $Vec\mathbb{H} \hookrightarrow Vec\mathbb{O}$, to (4;8)-brane. This involves the second fundamental geometry with an exceptional automorphism group. For self-dual (4;8)-brane we write

$$\tilde{P}^{a}_{\ \mu} = T\ \tfrac{1}{3!}\, e^{abcd}\ T_{\mu\nu\rho\sigma}\ X^{\nu}_{\ ,b} X^{\rho}_{\ ,c} X^{\sigma}_{\ ,d} \tag{4.1}$$

The completely antisymmetric tensor $T_{\mu\nu\rho\sigma}$ was introduced in [9]. It can be defined, making use of (3.7), from

$$\left.\begin{aligned} T_{0\mu\nu\rho} &= \pm C_{\mu\nu\rho} \\ T_{\mu\nu\rho\sigma} &= H_{\mu\nu\rho\sigma} \end{aligned}\right\} \tag{4.2}$$

where $\mu,\nu=1,2,\ldots,7$ but $\mu,\nu\neq 0$. Choosing the positive sign defines a self-dual tensor $T_{\mu\nu\rho\sigma}$ $\mu,\nu=0,1,2,\ldots,7$ while choosing the negative sign defines an anti-self-dual tensor. Changing between anti-clockwise and clockwise bases of $C_{\mu\nu\rho}$ has an equivalent effect so we can take the positive sign in (4.2) without loss provided we consider both clockwise and anti-clockwise bases.

Consider the 'double self-duality' equation analogous to self-dual Yang-Mills

$$F^{\mu\nu}_{\ ab} = \tfrac{1}{4}\ \varepsilon_{ab}^{\ \ cd}\ T^{\mu\nu}_{\ \ \rho\sigma}\ F^{\rho\sigma}_{\ cd} \tag{4.3}$$

where

$$F^{\mu\nu}_{\ ab} \equiv X^{\mu}_{\ ,[a} X^{\nu}_{\ ,b]} \tag{4.4}$$

Contract (4.3) with $X_{\nu}^{\ ,b}$ gives

$$X^{\mu}_{\ ,a} = \tfrac{1}{3!}\ \varepsilon_{a}^{\ bcd}\ T^{\mu}_{\ \nu\rho\sigma}\ X^{\nu}_{\ ,b} X^{\rho}_{\ ,c} X^{\sigma}_{\ ,d} \tag{4.5}$$

which is the self-dual equation arising from (4.1) and (1.4).

Contracting (4.5) with $X_{\mu}^{\ ,a}$ gives

$$\sqrt{g} = T_{\mu\nu\rho\sigma}\ X^{\mu}_{\ ,0} X^{\nu}_{\ ,1} X^{\rho}_{\ ,2} X^{\sigma}_{\ ,3} \tag{4.6}$$

which is the condition necessary for (4.1) to satisfy constraint (1.6). The equation of motion (1.5) and the second constraint (1.7) are automatically satisfied by (4.1) without further conditions.

5. (p,q) Quots

Consider a quaternionic curve satisfying

$$U^p + V^q = 0 \quad ; \quad U,V \in \mathbb{H} \tag{5.1}$$

A solution is given by

$$U = K^q \quad , \quad V = -K^p \quad ; \quad K \in \mathbb{H} \tag{5.2}$$

Take the case where

$$K = t + xi + yj + zk \equiv t + \underline{r} \tag{5.3}$$

and p=2, q=3, then

$$\left.\begin{aligned} U &= (t^3 - 3tr^2) + (3t^2 - r^2)\underline{r} \\ \text{and} \qquad V &= (r^2 - t^2) - 2t\underline{r} \end{aligned}\right\} \tag{5.4}$$

Call this quaternion analogy of a (2,3) torus knot, a (2,3) quot. Now construct an octonion X by catenating U to V, inserting a dimensionful constant (with dimensions of length), L, (which we shall henceforth set to 1) to avoid dimensional problems:

$$^{pq}X = (\, U \,,\, L^{q/p}\, V \,) \tag{5.5}$$

Then consider ^{pq}X as a potential solution of (4.5) or (4.6). Writing

$$^{pq}X^\mu = (\, ScU,\ VecU,\ ScV,\ VecV \,)$$

then

$$^{23}X^\mu = \left[\, (t^3 - 3r^2),\ (3t^2 - r^2)\underline{r},\ (r^2 - t^2),\ -2t\underline{r} \,\right] \tag{5.6}$$

Let us first compute $\sqrt{g}$ from this ^{23}X. We find

$$\sqrt{^{23}g} = (t^2 + r^2)\left[9(t^2 + r^2) + 4\right]\left[(3t^2 - r^2)^2 + (2t)^2\right] \tag{5.7}$$

For ^{23}X to have a chance of solving (4.6) Det $^{23}g_{ab}$ must be a perfect square. (5.7) shows that ^{23}X does satisfy this non-trivial criterion.

The left side of (4.6) does not appear to depend upon the basis chosen for octonion multiplication whereas the right side certainly does, from (4.2). Therefore ^{23}X could only be expected to satisfy (4.6) in one particular choice of basis. So what basis should we use?

In fact we have made an implicit choice of basis in our construction of ^{23}X. Defining a complex number as two real numbers catenated together is equivalent to saying that the first is the real part and the second is the imaginary part of the complex number. Symbolically;

$$\mathbb{C} \equiv (\mathbb{R}_1,\ \mathbb{R}_2) \equiv (\mathbb{R}_1 + i\mathbb{R}_2) \tag{5.8}$$

However, the situation is more involved in the case of quaternions because the real and imaginary parts are now themselves complex numbers.

$$\begin{aligned} \mathbb{H} \equiv (\mathbb{C}_1, \mathbb{C}_2) \equiv (\mathbb{C}_1 + j\mathbb{C}_2) &\equiv \left\{(\mathbb{R}_{11} + i\mathbb{R}_{12}) + j(\mathbb{R}_{21} + i\mathbb{R}_{22})\right\} \\ &= (\mathbb{R}_{11} + i\mathbb{R}_{12} + j\mathbb{R}_{21} - k\mathbb{R}_{22}) \end{aligned} \tag{5.9}$$

in the usual 123 basis. Note the appearance of the minus sign . To avoid this we shall put the basis element to the right of the coefficient. So our understanding of how we have derived an octonion from two quaternions is, symbolically,

$$\begin{aligned} \mathbb{O} \equiv (\mathbb{H}_1, \mathbb{H}_2) \equiv (\mathbb{H}_1 + \mathbb{H}_2 l) &\equiv \left\{(\mathbb{C}_{11} + \mathbb{C}_{12} j) + (\mathbb{C}_{21} + \mathbb{C}_{22} j)\right\} \\ &= \left\{\left[(\mathbb{R}_{111} + \mathbb{R}_{112} i) + (\mathbb{R}_{121} + \mathbb{R}_{122} i) j\right] + \left[(\mathbb{R}_{211} + \mathbb{R}_{212} i) + (\mathbb{R}_{221} + \mathbb{R}_{222} i)\right] j\right\} l \end{aligned} \tag{5.1}$$

It is this we are taking to be our octonion. Therefore we are implicitly taking $ij=k$ (i.e. 123), $il=m$ (i.e. 145), $jl=n$ (i.e. 246) and $(ij)l=kl=o$ (i.e. 347). The only admissible basis with those assignments is $\left|\begin{matrix}123\\246\\365\\451\\572\\617\\734\end{matrix}\right|$. Changing the signs of various coefficients of ^{23}X is equivalent to choosing different implicit bases.

Calculating the right hand side of (4.6) in this basis gives

$$\begin{aligned} ^{23}TXXXX &\equiv T_{\mu\nu\rho\sigma}\ {}^{23}X^\mu_{,0}\ {}^{23}X^\nu_{,1}\ {}^{23}X^\rho_{,2}\ {}^{23}X^\sigma_{,3} \\ &= (t^2 + r^2)\left[9(t^2 + r^2) + 4\right]\left[(3t^2 - r^2)^2 - (2t)^2\right] \end{aligned} \tag{5.11}$$

Comparing with (5.7) we see that the only difference is a single sign.

For the case of a (2,5) quot we have

$$ {}^{25}X = \left[(t^5 - 10t^3r^2 + 5tr^4),\ (5t^4 - 10t^2r^2 + r^4)\underline{r},\ (r^2 - t^2),\ -2t\underline{r}\right] \tag{5.12} $$

We find

$$ \sqrt{{}^{25}g} = (t^2 + r^2)\left[25(t^2 + r^2)^3 + 4\right]\left\{(5t^4 - 10t^2r^2 + r^4)^2 + (2t)^2\right\} \tag{5.13} $$

while, in the chosen basis,

$$ {}^{25}TXXXX = (t^2 + r^2)\left[25(t^2 + r^2)^3 + 4\right]\left\{(5t^4 - 10t^2r^2 + r^4)^2 - (2t)^2\right\} \tag{5.14} $$

These again only differ by a single sign. Similarily,

$$ \sqrt{{}^{94}g} = (t^2 + r^2)^2\left[16(t^2 + r^2)^2 + 9\right]\left\{\left[4t(t^2 - r^2)\right]^2 \oplus (3t^2 - r^2)^2\right\} \tag{5.15} $$

and

$$ \sqrt{{}^{35}g} = (t^2 + r^2)^2\left[16(t^2 + r^2) + 9\right]\left\{(5t^4 - 10t^2r^2 + r^4)^2 \oplus (3t^2 - r^2)^2\right\} \tag{5.16} $$

where $\oplus$ identifies the offending sign.

These results enable us to formulate the general empirical result that

$$ \sqrt{{}^{pq}g} = (t^2 + r^2)^{p-1}\left[q^2(t^2 + r^2)^{q-p} + p^2\right]\left\{ f(q)^2 + f(p)^2 \right\} \tag{5.17} $$

where

$$ f(n) = \sum_{j=1}^{\lceil n/2 \rceil} (-1)^{j+1} \binom{n}{2j-1} t^{n-2j+1}\, r^{2(j-1)} \tag{5.18} $$

The corresponding expression for ${}^{pq}TXXXX$ requires only the last + sign in (5.17) to be changed. Note that Det ${}^{pq}g_{ab}$ is always a perfect square. Changing, for example, the signature of the space-time metric destroys this necessary condition. ${}^{pp}TXXXX = 0$. Also note, if $f(p) = 0$ then ${}^{pq}X$ is a solution of (4.6).

The formulae of (5.17 & 18) enabled us to correctly predict

$$ \sqrt{{}^{27}g} = (t^2 + r^2)\left[49(t^2 + r^2)^5 + 4\right]\left\{(7t^6 - 35t^4r^2 + 21t^2r^4 - r^6)^2 \oplus (2t)^2\right\} \tag{5.19} $$

Figure 2: A representation of the (2,5) Quot of equation (5.6)

Obviously one would hope that changing to another basis, or aking the anti-self-dual version of $T_{\mu\nu\rho\sigma}$, or rearranging the ows and columns of $X^{\mu}{}_{,a}$, or multiplying some of the rows and olumns of $X^{\mu}{}_{,a}$ by -1, all of which leave $\sqrt{g}$ unchanged, would nable one to alter the offending sign in *TXXXX*. We argue that his is not possible. Firstly note that all of these variables are bsorbed in the 'change of basis' variable.

Take ${}^{23}X^{\mu}{}_{,a}$ and write it out explicitly.

$$
{}^{23}X^{\mu}{}_{,a} = \begin{bmatrix}
3t^2-3r^2 & -6tx & -6ty & -6tz \\
6tx & 3t^2-3x^2-y^2-z^2 & -2xy & -2xz \\
6ty & -2xy & 3t^2-x^2-3y^2-z^2 & -2yz \\
6tz & -2xz & -2yz & 3t^2-x^2-y^2-3z^2 \\
-2t & 2x & 2y & 2z \\
-2x & -2t & 0 & 0 \\
-2y & 0 & -2t & 0 \\
-2z & 0 & 0 & -2t
\end{bmatrix} \qquad (5.20)
$$

'XXXX picks one entry from each column for the non-zero components of $T_{\mu\nu\rho\sigma}$ and multiplies them together with a ±1 in front depending ipon the basis chosen. Notice that $\sqrt{{}^{23}g}$ in (5.7) contains terms ·16t^4, $+4x^6$, $+4y^6$ and $+4z^6$. There is only one way to obtain these erms in *TXXXX*:

$$+16t^4 \Rightarrow T_{5678} = +1 \qquad \therefore H_{4567} = +1 \qquad (5.21)$$

$$+4t^6 \Rightarrow T_{6534} = -1 \qquad \therefore H_{4523} = +1 \qquad (5.22)$$

$$+4y^6 \Rightarrow T_{7254} = -1 \qquad \therefore H_{4163} = +1 \qquad (5.23)$$

$$+4z^6 \Rightarrow T_{8235} = -1 \qquad \therefore H_{4127} = +1 \qquad (5.24)$$

Jsing (3.7), (5.21 & 22) imply that $C_{451} = C_{671} = C_{231}$ and (5.23 & 24) imply that $C_{415} = C_{635} = C_{275}$. We require an admissible basis to exist containing $\begin{vmatrix} 451 \\ 671 \\ 231 \\ 653 \\ 257 \\ \cdots \\ \cdots \end{vmatrix}$ which of necessity requires one of the unknown rows to be 437 or 473 because both missing rows must contain a 4 and the remaining four elements must be taken from 2,3,6 and 7 since every number has to appear three times in total. Now, rotating the rows into standard form in which there is no repetition in the columns, we find, fixing the first row as 145, only two possibilities, $\begin{vmatrix} 145 \\ 716 \\ 231 \\ 653 \\ 572 \\ \cdots \\ \cdots \end{vmatrix}$ and $\begin{vmatrix} 145 \\ 671 \\ 312 \\ 536 \\ 257 \\ \cdots \\ \cdots \end{vmatrix}$ both of which have 4,3 and 7 in the same column. Therefore it is impossible to add 437 or 473 as a new row without introducing repetition in the central column. Therefore no basis can make (5.11) equal to (5.7).

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